A simple dynamic model for pricing and hedging heterogenous CDOs

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Abstract

We present a simple bottom-up dynamic credit model that can be calibrated simultaneously to the market quotes on CDO tranches and individual CDSs constituting the credit portfolio. The model is most suitable for the purpose of evaluating the hedge ratios of CDO tranches with respect to the underlying credit names. Default intensities of individual assets are modeled as deterministic functions of time and the total number of defaults accumulated in the portfolio. To overcome numerical difficulties, we suggest a semi-analytic approximation that is justified by the large number of portfolio members. We calibrate the model to the recent market quotes on CDO tranches and individual CDSs and find the hedge ratios of tranches. Results are compared with those obtained within the static Gaussian Copula model.

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1 Introduction

One of the key challenges in the field of credit derivatives remains the construction of sophisticated mathematical models of Collateralized Debt Obligation (CDOs). The interest is motivated by the practical demand of tractable models capable of reliable, arbitrage-free valuation of portfolio loss derivatives, as well as calculation of sensitivities with respect to the underlying assets required for hedging purposes. Most CDO models that are currently used in practice are static, the industry standard being the Gaussian Copula (GC) model. In the static approach, the loss of the credit portfolio is modeled independently at each time step, disregarding the dynamics of the loss process.

Dynamic models suggested up to the present time can be loosely classified into two categories. Models that belong to the first category are constructed bottom up, such that the dynamics of the portfolio loss is built on top of default processes of underlying credit names. Default dynamics of each component, in tern, can be described either via structural approach originated by Merton (1974), or via reduced, intensity based approach where default events are unpredictable. A CDO model based on structural description of components was suggested by Inglis & Lipton (2007), see also Inglis *at el* (2008). The list of intensity based bottom-up CDO models is extensive, and here we refer only to a few publications which are related most closely to the present work: Duffie & Garleanu (2001), Duffie, Saita & Wang (2006), Eckner (2007), Feldhütter (2007), Frey & Backhaus (2007), and Mortensen (2006). The dynamic model recently suggested by Hull & White (2006) can also be attributed to the bottom-up framework, in spite of the fact that the authors assume the homogenous portfolio case at the initial stage of model construction. The main problem of the models belonging to this class is complexity: In order to achieve numerical feasibility, one is often forced to either choose an analytically tractable dynamics, thereby reducing calibration capability, or restrict considerations to the homogeneous portfolio case.

Models that belong to the second category postulate the dynamics for the portfolio loss directly, not referring to the underlying portfolio members. This framework was first suggested in works of Giesecke & Goldberg (2005), Schönbucher (2005), and Sidenius *at el* (2005). Other models that were recently developed within this framework can be found in works by Arnsdorf & Halperin (2007), Brigo *et al* (2006), Errais *et al* (2006), Ding *et al* (2006), and Lopatin & Misirpashaev (2008). The top-down framework allows reducing the dimensionality of the problem, making it possible to formulate a numerically tractable model having sufficient flexibility for calibration to the market of CDO tranches. Models of this kind are natural candidates for pricing the dynamics-sensitive derivatives of the portfolio loss. This approach, however, obscures the relationship of the model to the underlying portfolio members, and requires special techniques to recover the dynamics of single-name intensities. Such a procedure, the so-called random thinning, was suggested in the work of Giesecke & Goldberg (2005). [See also the recent paper by Halperin & Tomecek (2008)]. More detailed review of the two approaches can be found in the recent article by Giesecke (2008).

In the present work, we suggest a simple dynamic bottom-up model having sufficient flexibility for simultaneous calibration to the market of credit default swaps (CDSs) and CDO tranches, and capable of evaluating tranche hedge ratios as well as simple instruments whose values can be expressed through a one-dimensional marginal distribution of the portfolio loss. The model is defined by specifying the form of the default intensity for each underlying asset as a function of time, t, and the number of defaults, $N(t)$, accumulated in the credit portfolio up to t. Our choice of the state variable can be better understood on examination of the simplest model within the top-down framework. This model, which in analogy with the local volatility model is sometimes called the local intensity model, represents the one-dimensional Markov chain, where default transitions $N \to N+1$ are governed by intensity, $\lambda(N(t),t)$, which is a deterministic function of N and t. Remarkably, it can be shown that any top-down credit model with default intensity given by an adaptive stochastic process of a general kind can be brought to the form of the local intensity model as long as instruments under consideration can be expressed through the one-dimensional marginal distribution of the portfolio loss [31]. Thus, a "minimal" multiname dynamic model can be constructed as an extension of the local intensity model by choosing the default intensity of each name to be a deterministic function of $N(t)$.

Our model can also be viewed as a special case of the default contagion model [9]. In the default contagion framework, the default intensity of the $k-$th asset is given by the deterministic function $\lambda_k(\mathbf{n}(t),t)$ of the portfolio state represented by the vector of asset default indicators $\mathbf{n} = (n_1, n_2, ..., n_{N_0})$, N_0 being the number of assets in the portfolio. Default of the $k-$th asset thereby affects (typically increasing) the default intensities of other assets. In its general form, the default contagion model contains a huge number of free parameters and, in practice, one deals with default intensities of a particular functional form. Examples of specific models that belong to this framework can be found in papers by Davis & Lo (2001), Jarrow & Yu (2001), and Laurent *et al* (2007). Our model corresponds to the special case where each default intensity depends on the total number of defaults $N = n_1 + n_2 + ... + n_{N_0}$.

Default contagion models are Markovian and, thus, they support the application of many standard numerical techniques, including forward/backward induction and Monte Carlo simulations. However, the huge size of the configuration space, 2^{N_0}, makes application of the forward/backward induction unfeasible. Construction of an effective Monte Carlo scheme is also a difficult task because of the requirement of calibration to tranches and individual credits. We resolve this difficulty by developing a semi-analytic approach that makes use of the fact that the number of portfolio members is large. This technique allows reducing the original multidimensional problem to the system of N_0 coupled one-dimensional Focker-Plank equations. Further, we show that, within the developed forward induction scheme, the calibration to individual CDSs can be implemented automatically with essentially no extra computational cost. Calibration to tranches is implemented as a mixture of bootstrap and multidimensional solving procedures similar to the scheme used by Arnsdorf & Halperin (2007). As a result, the complete calibration procedure becomes computationally effective and, for typical market data, takes less than a minute on a laptop with an AMD Turion ML-40 (2.19 GHz) processor.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we describe the semi-analytic approach to solving the model. In Section 4, we present the procedure of model calibration, along with numerical results for the fit to recent market data. Tranche hedge ratios are considered in Section 5, where results of the dynamic model are compared with those obtained within the GC model. In Section 6, we extend our approach to the case of a portfolio with heterogeneous recovery coefficients. In Section 7, we discuss the connection of the default contagion framework to the double stochastic one in general terms. In Section 8 we include the effect of negative correlations between the recovery coefficients and default intensity. We conclude in Section 9.

2 Model

Under the standard assumption of constant recovery coefficients, stripping each CDS allows reproducing the survival probabilities of individual portfolio members. This procedure relies on time interpolation of the implied default intensity, which, in the simplest case, is taken to be piece constant. Thus, we assume that the $k-$th portfolio member is characterized by the survival probability, $p_k(t)$, and the recovery coefficient, R_k. We postulate the default intensity of the $k-$th asset to be

$$\lambda_k(t) = a_k(t) + b_k(t)\, Y(N(t), t), \tag{1}$$

where $N(t)$ is the number of defaults accumulated in the portfolio up to time t. The function $Y(N, t)$ is the common (systemic) component that serves to model the correlations of default events. The functions $a_k(t)$ and $b_k(t)$ are specific to the $k-$th asset. For a given $Y(N, t)$, functions $a_k(t)$ and $b_k(t)$ are used to fit the model to the survival curves of individual assets. The function $Y(N, t)$ provides a freedom for calibrating the model to CDO tranches. It is clear, however, that the model (1) has too many free parameters (in addition to the freedom in interpolation of the function $Y(N, t)$ which will be discussed below) since both functions $a_k(t)$ and $b_k(t)$ cannot be defined uniquely by fitting a single survival curve. For this reason, below, we choose a particular specification of model (1) containing no extra parametric freedom. We take $a_k(t) = 0$, such that the intensity of the $k-$th asset becomes a product of the idiosyncratic component $b_k(t)$ and the systemic component, $Y(N, t)$. An alternative simple choice would be to take $b_k(t) = 1$ such that each single asset intensity would be a sum of the idiosyncratic component $a_k(t)$ and systemic component, $Y(N, t)$. We prefer the first choice because it guarantees the positivity of all default intensities by construction. Nonzero values of $a_k(t)$, however, will be used later for calculating the tranche hedge ratios. In Section 7, we comment on restrictions in applications of model (1) and compare it with models of the double stochastic framework.

3 Semi-analytic approach

The model (1) could be treated directly by writing down the multidimensional Focker-Planck equation on the full distribution function of defaults and then solving it numerically. This way, however, is practically unfeasible, since the size of the configuration space of the full distribution function is given by 2^{N_0} which, for a typical number of assets, $N_0 = 125$, exceeds the operating memory of modern computers by orders of magnitude. The simulation approach is certainly a possible alternative. However, one has to keep in mind that the model must be calibrated (at least) to the market of CDO tranches via iterative solving, and this, most likely, will make the simulation method unfeasible, too. Below, we present a semi-analytic approximate scheme to solving model (1) which is justified by a large number of assets in the credit basket. Formally, this approach becomes exact in the limit $N_0 \to \infty$.

Let us consider the probability that the $k-$th asset has survived up to time t, and that, at the same time, N assets in total have defaulted,

$$P_k(N, t) = \mathrm{P}[\, n_k(t) = 0, N(t) = N\,]. \tag{2}$$

Here, n_k is the default indicator of the $k-$th asset ($n_k = 1$ assumes default and $n_k = 0$ survival) and $N(t)$ is the number of defaults in the basket accumulated up to time t. We note that $P_k(N_0, t) = 0$. The probability $P_k(N, t)$ satisfies the following Focker-Planck equation,

$$\frac{d}{dt} P_k(N, t) = \Lambda_k(N-1, t) P_k(N-1, t) - \left[\Lambda_k(N, t) + \lambda_k(N, t)\right] P_k(N, t), \qquad (3)$$

where

$$\lambda_k(N, t) = a_k(t) + b_k(t)\, Y(N, t), \qquad (4)$$

and $\Lambda_k(t)$ is the intensity of any asset other than the $k-$th to default, conditioned that the total number of defaults in the basket is N, and that the $k-$th asset has survived,

$$\Lambda_k(N, t) = \frac{1}{\delta_t} \sum_{p \neq k} \mathrm{E}[\, n_p(t + \delta_t) - n_p(t) \,|\, n_k(t) = 0,\, N(t) = N\,], \qquad \delta_t \to 0. \qquad (5)$$

Instead of writing the multiplier $\mathbf{1}_{N>0}$ in front of the first term in the right hand side of Eq. (3), we define that $P_k(N, t) = 0$ at $N < 0$. We follow this convention throughout the paper for the sake of brevity. The derivation of Eq. (3) is presented in Appendix A. To gain an intuitive understanding of this equation, it is instructive to imagine that the $k-$th asset was "extracted" from the basket such that a system consisting of a single asset and the basket with remaining $N - 1$ assets is formed. The meaning of the terms in Eq.(3) then becomes straightforward: Default transitions in the reduced basket are described via the intensity $\Lambda_k(N, t)$, while the term containing $\lambda_k(N, t)$ is responsible for a possible default of the k-th asset itself. The boundary condition for Eq. (3) at the origin $t = 0$ is

$$P_k(N, 0) = \begin{cases} 1, & N = 0, \\ 0, & N \neq 0. \end{cases} \qquad (6)$$

The intensity of defaults in the basket having all but $k-$th assets, $\Lambda_k(N, t)$, is unknown. Direct determination of this quantity is difficult and, in the case of a realistic CDO having 125 or even more members, would require an application of simulations. It is clear, however, that, in the case of a large portfolio, $\Lambda_k(N, t)$ is approximately given by the intensity of any asset in the basket to default conditioned on the total number of defaults. The latter quantity is formally defined as

$$\Lambda_B(N, t) = \frac{1}{\delta_t} \sum_{k} \mathrm{E}[\, n_k(t + \delta_t) - n_k(t) \,|\, N(t) = N], \qquad \delta_t \to 0. \qquad (7)$$

The approximation $\Lambda_k(N, t) \approx \Lambda_B(N, t)$ can be improved by multiplying the basket intensity by $1 - 1/(N_0 - N)$; this ensures that the scheme becomes exact in the homogenous portfolio case. Thus, we take $\Lambda_k(N, t)$ in the form

$$\Lambda_k(N, t) = (1 - \frac{1}{N_0 - N})\, \Lambda_B(N, t). \qquad (8)$$

In the next section, we will show that correction $1/(N_0 - N)$ plays an important role in making the scheme internally consistent and will discuss possible alternatives to Eq. (8).

To complete the system of Eqs. (1, 3, 8), one needs to find the basket intensity $\Lambda_B(N, t)$. To fulfill this goal, we will first show that the probability distribution of the defaults in the basket, $P_B(N, t)$, is related to the probabilities $P_k(N, t)$ via

$$P_B(N, t) = \frac{1}{N_0 - N} \sum_{k=1}^{N_0} P_k(N, t). \qquad (9)$$

This equation can be proven by presenting the basket default distribution function, $P_B(N,t)$, in the form

$$P_B(N,t) = \sum_{\mathbf{n}} p(\mathbf{n},t)\ \mathbf{1}_{n_1+n_2+...+n_{N_0}=N}, \tag{10}$$

where $p(\mathbf{n},t)$ is the probability density of the portfolio state represented by the vector of default indicators $\mathbf{n} = (n_1, n_2, ..., n_{N0})$, and where summation goes over all possible portfolio states. Similarly, one can write

$$P_k(N,t) = \sum_{\mathbf{n}} p(\mathbf{n},t)\ \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+n_2+...+n_{N_0}=N}. \tag{11}$$

Equation (9) can be proven by inserting Eq. (11) into its r.h.s. and further term by term comparison with Eq. (10).

The basket intensity $\Lambda_B(N,t)$ now can be found by considering the expected number of default transitions, $N \to N+1$, taking place in the whole basket within the time interval $t, t+\delta_t$: This quantity, on the one hand, is given by $\Lambda_B(N,t)P_B(N,t)\,\delta_t$; on the other hand, it is $\sum_k \lambda_k(N,t)P_k(N,t)\,\delta_t$. Equating the two results, we obtain

$$\Lambda_B(N,t) = \frac{\sum\limits_{k=1}^{N_0} \lambda_k(N,t)P_k(N,t)}{P_B(N,t)}. \tag{12}$$

The system of equations (3, 4, 8, 9, 12), along with the boundary condition (6), define uniquely the functions $p_k(N,t)$, $P_B(N,t)$, and $\Lambda_B(N,t)$. Indeed, assuming that the single-name intensities $\lambda_k(N,t)$ are given, one can integrate Eq. (3) simultaneously for all k using $\Lambda_k(N,t)$ defined through Eqs. (8, 9, 12) on each integration time step.

3.1 Self consistency check

The functions $P_B(N,t)$ and $\Lambda_B(N,t)$ that represent, respectively, the probability density of defaults, and the intensity of a default of any asset in the basket conditioned that $N(t) = N$, play an important role in formulation of the semi-analytic scheme developed in the previous section. Because of their meanings, these two functions must be related by the following Focker-Planck equation,

$$\frac{d}{dt}P_B(N,t) = \Lambda_B(N-1,t)P_B(N-1,t) - \Lambda_B(N,t)P_B(N,t). \tag{13}$$

This equation was neither explicitly nor implicitly used in the formulation of the semi-analytic scheme and, thus, it is important to be sure that functions $P_B(N,t)$ and $\Lambda_B(N,t)$ obtained within this scheme satisfy Eq. (13).

Here, we show that this equation is satisfied indeed. Under approximation (8), $\Lambda_k(N,t)$ is independent of k. Summing both parts of Eq. (3) over k and using Eqs. (9, 12), one can bring it to the form

$$\frac{d}{dt}P_B(N,t) = \frac{N_s+1}{N_s}\Lambda_k(N-1,t)P_B(N-1,t) - \left(\Lambda_k(N,t) + \frac{\Lambda_B(N,t)}{N_s}\right)P_B(N,t), \tag{14}$$

where $N_s = N_0 - N$ is the number of survived assets in the basket. Now, using Eq. (8) for $\Lambda_k(N,t)$, one can see that Eq. (14) is consistent with Eq. (13).

Note that Eq. (14) would not be satisfied if the term $1/(N_0 - N)$ in Eq. (8) were neglected. This correction, thus, plays an important role in ensuring internal consistency of the scheme. The form of this correction, however, is not unique. For example, one can show that the approximation

$$\Lambda_k(N,t) = \Lambda_B(N,t) - \lambda_k(N,t) \tag{15}$$

also leads to the internally consistent scheme. In Appendix A, we consider the question of consistency in more general terms.

4 Model calibration

The model (1) has to be calibrated to the market data on CDO tranches and individual CDSs. Recall that the single-name spreads can be fitted by choosing functions $b_k(t)$, while the spreads of CDO tranches are adjusted via the function $Y(N,t)$. Below, we show that calibration to single-name survival curves can be performed exactly and essentially with no extra computational cost during the numerical integration of Eq. (3). Calibration to CDO tranches represents a mixture of bootstrap and iterative fitting. The latter procedure is similar to calibration of the local intensity model to CDO tranches, with function $Y(N,t)$ playing a role similar to that of the local intensity [7, 31]. In this section, we will first assume that the function $Y(N,t)$ is given and describe the calibration to the single-name survival probabilities. Then, we will turn to calibration of the function $Y(N,t)$ to CDO tranches.

4.1 Calibration to the asset survival probabilities

We will consider the calibration procedure within the simplest, first order in time scheme. Fixing the set of times $t_0, t_1, ..., t_M$, where t_0 is the start time and t_M is the horizon, we write Eq. (3) in the form

$$\frac{P_k(N,t_{i+1}) - P_k(N,t_i)}{\Delta_i} = \Lambda_k(N-1,t_i)P_k(N-1,t_i) - \left[\Lambda_k(N,t_i) + \lambda_k(N,t_i)\right] P_k(N,t_i), \tag{16}$$

where $\Delta_i = t_{i+1} - t_i$. The consistency with the market data on single-name CDS can be ensured by requiring that the single-name survival probabilities, $p_k(t)$, can be reproduced within the model

$$p_k(t_i) = \sum_{N=0}^{N_0-1} P_k(N,t_i). \tag{17}$$

The above equation must be satisfied on each integration time step by choosing the proper values of $b_k(t_i)$. This can be done with the help of equation

$$\lambda_k(t_i) = \frac{1}{p_k(t_i)} \sum_{N=0}^{N_0-1} \lambda_k(N,t_i)\, P_k(N,t_i), \tag{18}$$

where $\lambda_k(t)$ is the marginal default intensity of the $k-$th asset implied by the survival curve

$$\lambda_k(t_i) = \frac{1}{p_k(t_i)} \frac{p_k(t_i) - p_k(t_{i+1})}{\Delta_i}. \tag{19}$$

Equation (18) can be obtained via the summation of both parts of Eq. (16) over N, from $N = 0$ up to $N_0 - 1$, and has a straightforward meaning: It equals the marginal default intensity of the $k-$th asset calculated from the survival curve (l.h.s.) with that obtained from the model. Now, inserting Eq. (4) into Eq. (18) and resolving the obtained equation with respect to $b_k(t)$, one obtains

$$b_k(t_i) = \lambda_k(t_i)\, p_k(t_i) \left(\sum_{N=0}^{N_0-1} Y(N,t_i)\, P_k(N,t_i) \right)^{-1}. \qquad (20)$$

The procedure of model calibration to CDSs can be summarized as follows:

1. Begin with initial date $i = 0$ and set the values of $P_k(N,0)$ according to Eq. (6).
2. Find the values of $b_k(N,t_i)$ according to Eq. (20).
3. Find the values of $\lambda_k(N,t)$, $P_B(N,t)$, $\Lambda_B(N,t)$, and $\Lambda_k(N,t)$ according to Eqs. (4, 9, 12, 8), respectively.
4. Find the values of $P_k(N,t)$ at the next time node according to Eq. (16).
5. Set $i \to i+1$ and repeat the steps 2–5 until the horizon is reached.

4.2 Calibration to tranches

We will consider the calibration procedure to CDO tranches in a simplified case of a portfolio with homogeneous recovery coefficients, $R_k = R$, and notionals, $A_k = A$. This allows expressing the portfolio loss through the number of defaults as $L = hN$, where $h = A(1-R)$ is the loss given default. The value of a CDO tranche in this case can be computed directly from the probability density of the number of defaults, $P_B(N,t)$. Generalization of our approach to a fully heterogeneous basket is presented below in Section 6.

Market data on CDO tranches is not complete for uniquely determining the function $Y(N,t)$ and, thus, to proceed, one needs to assume that $Y(N,t)$ has a certain functional form. We will parameterize the $Y(N,t)$ surface via the linear interpolation in loss between the values $Y_k(t)$ defined at the loss levels $l_p(t)$:

$$Y(N,t) = Y_{p-1}(t)\, \frac{l_p(t) - Nh}{l_p(t) - l_{p-1}(t)} + Y_p(t)\, \frac{Nh - l_{p-1}(t)}{l_p(t) - l_{p-1}(t)}, \qquad l_{p-1} < Nh < l_p. \qquad (21)$$

In the simplest case, the functions $l_k(t)$ can be set to be losses corresponding to the detachment levels of tranches,

$$l_p(t) = L_p. \qquad (22)$$

We will also assume that functions $Y_k(t)$ are time independent within the intervals between the tranche maturities:

$$Y_p(t) = Y_{i,p}, \qquad T_{i-1} \le t < T_i, \qquad (23)$$

where $T_1, T_2, \ldots, T_M$ is the set of tranche maturity dates and $T_0 = t_0$ is the start date. This form allows setting up a bootstrap calibration to CDO tranches with different maturities: First, one calibrates the model to tranches with maturity T_1 setting $Y_{1,p}$. Then, the model is calibrated to tranches that mature at T_2, adjusting $Y_{2,p}$, and keeping $Y_{1,p}$ unchanged. The procedure is repeated until the model is fully calibrated. This parametrization is similar to the form of the local intensity used in the work [7]. In the top-down framework, this scheme usually provides a good fit to the tranches for most of the market data before the credit crisis. Our numerical experiments showed that similar calibration quality is achieved in the multiname model for heterogenous CDO portfolios.

However, the above described scheme often fails in fitting after-crisis tranche quotes with acceptable accuracy. Typically, the model significantly overprices the senior tranche and underprices the supersenior tranche at 5-year maturity. We found that the assumption of time independence of the function $Y(N,t)$ within the intervals between adjacent tranche maturities is one of the key reasons for this failure. This assumption can be justified only by the smallness of the time intervals between the tranche maturity dates. In the case of the standard set of tranche maturities, $5, 7, 10$ years, it could be justified for the two last intervals, but it hardly holds for the first one. Indeed, when the time evolution just begins, the probability distribution function of losses is confined to a region of low losses lying within the equity tranche. At these times, only $Y_{1,1}$ component, corresponding to the detachment level of the equity tranche, can affect the dynamics. The value $Y_{1,6}$, corresponding to the supersenior tranche, on the contrary begins to affect the dynamics only at later evolution stages, when the portfolio loss can reach the supersenior tranche with significant probability. This illustrates that values of Y corresponding to more senior tranches effectively have lower leverage for guiding the dynamics. To resolve this problem, we suggest scaling the loss levels l_k linearly in time for the first interval as

$$l_p(t) = L_p \,\frac{t - t_0}{T_1 - t_0}, \qquad t_0 < t < T_1. \tag{24}$$

Our numerical experiments showed that this modification results in a dramatic improvement of the calibration quality for all considered data. All results presented below assume that scaling (24) was used.

4.3 Numerical results on calibration

We present numerical data on calibration to the sets of iTraxx Series 9 tranches maturing at 5, 7, 10 years and quoted on April 17, 2008. The index was quoted for maturities 3, 5, 7, 10 years with the corresponding spreads 65.8, 89.8, 95, 98.8 *bp*. Stripping of the single-name quotes was done assuming the piece-constant implied marginal default intensity. The market spreads for each CDS were taken at 5, 7, 10 years, skipping the quotes at lower maturities even if they were available. To avoid a possible arbitrage between the index and single-name spreads, the survival curves were matched to index quotes via homogeneous rescaling of the obtained survival curves. This procedure is standard to all bottom-up credit models.

Along with the heterogeneous portfolio with market-given CDS spreads, we also consider an artificial homogenous situation where all CDSs are set to index quotes. Results of the fits to CDO tranches are presented in Tables 1 and 2 for homogenous and heterogenous situations, respectively. In both cases, the results of the fit are well within the market bid-offer spreads. Using other data, we generally have observed that calibration quality is essentially not sensitive to the heterogeneity of the portfolio. The basket default intensity, $\Lambda_B(N,t)$, in the heterogeneous portfolio case is presented in Figure 1.

5 Hedging a CDO tranche

In this section, we consider the hedging of a CDO tranche against the market movements of spreads of the underlying CDSs. This question is usually addressed by defining the amount of protection, delta, that needs to be bought on each credit name in order to hedge a long position in a CDO tranche. Delta of the $k-$th credit is defined as the ratio of the change in the tranche value, $\Delta V^{\rm tr}$, and the change in the value of $k-$th CDS, $\Delta V_k^{\rm cds}$, occurring under a small shift of the $k-$th CDS spread:

$$\delta_k = \frac{\Delta V^{\rm tr}}{\Delta V_k^{\rm cds}}. \tag{25}$$

This approach to hedging of the spread risk is similar to vega hedging in the Black-Scholes model. While it can hardly be rigorously justified, it is robust, and now widely used in practice. Here we follow this standard, simplified scheme. Recent achievements in the rigorous approach to hedging of a credit derivative with respect to different kind of risks can be found in the work by Bielecki *et all* (2008) (see also references therein).

5.1 Hedge ratios in the Gaussian Copula model

At present, the tranche deltas are most often calculated via the Gaussian Copula (GC) model. This model is also frequently used to gain an intuitive understanding of the dependence of hedge ratios on tranche subordination, contract maturity, spreads of the underlying credits, etc. Because of these reasons, below, we will compare deltas obtained within the dynamic model with corresponding results of the GC model.

Recall that the GC model is defined through the set of correlated Gaussian random variables

$$X_k = \sqrt{1-\beta_k^2}\, \xi_k + \beta_k\, Z, \tag{26}$$

where ξ_k and Z are uncorrelated univaried Gaussian random variables and the factor loadings, β_k, define the correlation matrix of variables X_k. At time t, the default of the $k-$th asset is assumed to occur if the random variable X_k falls below the barrier $c_k(t)$. The barrier of the $k-$th CDS can be set via matching the default probability of the $k-$th asset according to equation

$$N(c_k(t)) = p_k^{\rm d}(t), \tag{27}$$

where $N(x)$ is the cumulative normal distribution and $p_k^{\rm d}(t)$ is the probability of the $k-$th asset to default before time t. This way the GC model is calibrated to portfolio credits. To achieve a stable calibration to CDO tranches, it is common to use the so-called base correlation scheme. In this approach, the value of a tranche with attachment points (k_1, k_2) is presented as the difference of the values of two base tranches with attachment levels $(0, k_2)$ and $(0, k_1)$, respectively, which are taken at different correlations. Calibrating the tranches consequently, in accordance with their seniority, one obtains the so-called base correlation curve. The base correlation approach is obviously not intrinsically consistent, which is the main drawback of the GC model (leaving aside fundamental drawbacks of the static approach).

To find the delta of a tranche with respect to the $k-$th credit in the GC model, one shifts the $k-$th spread by a small amount (typically 1 *bp*) by readjusting accordingly the default probabilities at the relevant dates t_i. The barriers $c_k(t_i)$ are then reset according to Eq. (27), the tranche value is recalculated, and delta is obtained from Eq. (25).

It is important to note that this procedure assumes the hedging of a tranche against the idiosyncratic change of the default probability of a given CDS. Indeed, shifting the barrier of the $k-$th asset does not affect in any way the distribution of defaults of other assets in the portfolio. Formally, this can be written as

$$\delta_k \sum_{n_k=0,1} p(\mathbf{n}, t) = 0, \tag{28}$$

where δ_k denotes the variation corresponding to the shift of the barrier of the $k-$th asset.

5.2 Hedge ratios in the dynamic model

The recipe presented above for the calculation of the hedge ratios is to some extent uncertain because it does not prescribe which model parameters should be adjusted and which should stay constant when taking the derivative in Eq. (25). Calculation within the GC model, for example, is based on the intuitively natural assumption that base correlations are not perturbed. For the static factor models, this problem was discussed in Ref. [3]. The same uncertainty is present in the dynamic approach. To use Eq. (25), one has to identify the model parameters that, similar to correlation coefficients in the GC model, should stay unperturbed when taking the derivative in Eq. (25). We will identify the default correlations with the jumps in the intensities, λ_k, taking place at the default of an asset in the basket. Since, in our model, the individual intensities depend only on the number of defaults, N, and time, t, these jumps can be written as

$$\kappa_k(N, t) = \lambda_k(N+1, t) - \lambda_k(N, t). \tag{29}$$

Thus, we postulate that the tranche delta with respect to the $k-$th credit is given by Eq. (25), where the derivative is taken under the constraint that the surface $\kappa_k(N, t)$ is kept unchanged. Given the expression for the default intensities (1), this assumes that functions $a_k(t)$ can be used for adjusting the model to the shifted single name spreads, while the systemic intensity $Y(N, t)$ and factors $b_k(t)$ should be kept unperturbed. The calibration procedure with adjusting coefficients $a_k(t)$ is very similar to the one described in Section 4 with the difference that Eq. (18) must be resolved with respect to the coefficients $a_k(t)$ which were kept zero. Inserting Eq. (4) into Eq. (18) and using Eq. (17), one obtains

$$a_k(t_i) = \lambda_k(t_i) - \frac{b_k(t_i)}{p_k(t_i)} \sum_{N=0}^{N_0-1} Y(N, t_i)\, P_k(N, t_i). \tag{30}$$

The described procedure for calculation of the tranche delta with respect to the $k-$th asset can be summarized as follows:

1. Calibrate the dynamic model to CDO tranches and single-name spreads following the procedure of Section 4.
2. Find the perturbed values of the survival probability and default intensity of the $k-$th asset corresponding to a 1 *bp* shift of the spread.
3. Recalibrate the model keeping the functions $b_k(t)$ and $Y(N, t)$ unchanged (as obtained on Step 1) and adjusting functions $a_k(t)$ according to Eq. (30).

4. Obtain tranche delta via Eq. (25) with ΔV^{tr} being the differences of the tranche values obtained on Steps 3 and 1.

We note that, while conceptually this procedure looks similar to the one used in the GC model, there is a significant qualitative difference between the two approaches: The GC model assumes that hedging ratios are calculated with condition (28) being satisfied, meaning that the change of the single-asset default probability does not in any way affect other portfolio assets. This condition does not hold for the suggested dynamic definition of the hedge ratios. Changing the survival probability of a given asset, one immediately perturbs the default intensities of all other assets. When presenting our numerical results, we will refer to the procedure described above as "contagious".

5.3 Recovering idiosyncratic risk in the dynamic model

Ignorance of the effect of default contagion is one of the drawbacks of the static GC approach. Indeed, the fact that the market fluctuations of single-name spreads are correlated can hardly be doubted and, therefore, the hedge ratios obtained in the idiosyncratic manner should be taken with caution. Yet, the idiosyncratic risk provides useful additional information and corresponding hedge ratios can also be of interest to practitioners. Below, we present a recipe for deriving tranche deltas within the developed dynamical approach which is consistent with condition (28).

In the case of homogeneous recovery coefficients, the value of a CDO tranche is uniquely defined by the basket probability distribution $P_B(N,t)$. Therefore, to determine the tranche delta, it is enough to find the change in the basket probability density, $\delta_k P_B(N,t)$, under the idiosyncratic change of the survival probability of the $k-$th asset. It turns out that $\delta_k P_B(N,t)$ can be expressed through the corresponding change of the probability density $P_k(N,t)$ defined by Eq. (2) as

$$\delta_k P_B(N,t) = \delta_k(P_k(N,t) - P_k(N-1,t)). \tag{31}$$

This expression can be proven as follows: Let us present the basket probability density as

$$P_B(N,t) = P_k(N,t) + P_k^d(N,t), \tag{32}$$

where $P_k^d(N,t)$ is the probability that up to time t the $k-$th asset defaulted and that the total number of defaults in the basket is N. Equation (32) has an obvious meaning: the probabilities for the $k-$th asset to survive and to default conditioned on the total number of defaults must sum to one. Thus, the change in the basket probability density can be written as

$$\delta_k P_B(N,t) = \delta_k P_k(N,t) + \delta_k P_k^d(N,t). \tag{33}$$

But, according to Eq. (28), under the idiosyncratic change of the survival probability of the $k-$th asset,

$$\delta_k(P_k(N,t) + P_k^d(N+1,t)) = 0. \tag{34}$$

Equation (31) now follows immediately from equations (33) and (34).

Thus, the problem of finding the change of the tranche value under the idiosyncratic perturbation of the survival probability of the $k-$th asset is reduced to finding $\delta_k P_k(N,t)$ in Eq. (31).

According to the Focker-Planck equation (3), the probability density $P_k(N,t)$ is defined uniquely by the conditioned default intensity of the $k-$th asset, $\lambda_k(N,t)$, and by the conditioned intensity of defaults in the basket having all but $k-$th asset, $\Lambda_k(N,t)$. It is important to note that the latter, by definition, should be kept unperturbed when finding the delta as long as one deals with the idiosyncratic risk. One could think that this condition is not consistent with perturbing the single asset default intensity, $\lambda_k(N,t)$, because of the effect of default contagion. Fortunately, this is not the case, and intrinsic consistency of the presented scheme is not violated. Indeed, $\Lambda_k(N,t)$ is defined with the condition that $k-$th asset has survived and, prior to the default of the $k-$th asset, the information on the change of the intensity $\lambda_k(N,t)$ is not accessible to other assets.

Finally, we have to specify the change of the single asset default intensity. We choose to rescale the default intensity of the k-th asset,

$$\lambda_k(N,t) \to \lambda'_k(N,t) = \kappa(t)\lambda_k(N,t). \tag{35}$$

This definition is preferable to changing some model parameters in a specific way because it can be applied to any intensity-based bottom-up model. A similar procedure was used in Ref. [14]. Within the first order in time scheme, the probability of $P_k(N,t)$ is determined by Eq. (16). The perturbed probability $P'_k(N,t) = P_k(N,t) + \delta_k P_k(N,t)$ is, thus, defined by

$$\frac{P'_k(N,t_{i+1}) - P'_k(N,t_i)}{\Delta_i} = \Lambda_k(N-1,t_i)P'_k(N-1,t_i) - \left(\Lambda_k(N,t_i) + \kappa(t)\lambda_k(N,t_i)\right) P'_k(N,t_i). \tag{36}$$

The scaling function $\kappa(t)$ can be found by matching the perturbed survival probability of the $k-$th asset using the procedure described in Section 4: Summing both parts of Eq. (36) over N from 0 to $N_0 - 1$, we obtain

$$\lambda'_k(t_i) = \frac{\kappa(t_i)}{p'_k(t_i)} \sum_{N=0}^{N_0-1} \lambda_k(N,t_i)\, P'_k(N,t_i). \tag{37}$$

Here, $p'_k(t)$ is the perturbed marginal survival probability of the $k-$th asset and $\lambda'_k(t_i)$ is the corresponding marginal default intensity $\lambda'_k(t_i) = (p'_k(t_i) - p'_k(t_{i+1}))/p'_k(t_i)\Delta_i$. The scaling function $\kappa(t)$ now can be found from Eq. (37):

$$\kappa(t) = p'_k(t_i)\lambda'_k(t_i) \left(\sum_{N=0}^{N_0-1} \lambda_k(N,t_i)\, P'_k(N,t_i) \right)^{-1}. \tag{38}$$

Equation (36) can be solved numerically by going forward in time and taking the values of $\kappa(t_i)$ from Eq. (38) on each time step. For clarity, below, we summarize the calculation of the delta of the $k-$th credit:

1. Calibrate the dynamic model to CDO tranches and single-name spreads following the procedure of Section 4.
2. Find the perturbed values of the survival probability and default intensity of the $k-$th asset ($p'_k(t)$ and $\lambda'_k(t_i)$, respectively) in accordance with the shifted spread of the $k-$th asset.

3. Find P_k' by solving Eqs. (36, 38) forward in time.
4. Find the change in the basket probability density according to Eq. (31), calculate the corresponding change in the tranche values, and obtain delta from Eq. (25).

This procedure will be further referred to as “idiosyncratic”.

5.4 Numerical results on tranche deltas

Here, we present the numerical results on tranche deltas. We use the same market data as in Section 4.3 and calculate deltas for all available tranches with respect to CDSs that mature at the same dates with tranches. As in Section 4.3, along with the real market data on CDSs, we also consider an artificially homogeneous setup where all CDS spreads are set to corresponding index spreads.

Tranche deltas calculated within the two dynamic methods are presented in Tables 3 and 4 for homogeneous and heterogeneous cases, respectively, along with the values obtained via the GC model. In the heterogeneous case, we present our results for three representative portfolio members with low, moderate, and high values of the spreads.

We begin with the discussion of the homogenous case. Comparing results obtained within the two dynamic methods, one can see that deltas of senior tranches obtained within the first (contagious) method are higher than those obtained within the second (idiosyncratic) one. This effect disappears with lowering the tranche seniority and turns to the opposite for the equity tranche. Such behavior is expected since the clustering of defaults, disregarded by the idiosyncratic approach, causes senior tranches to be more risky.

Tranche deltas obtained within the contagious dynamic approach turn out to be pretty close to the results obtained within the GC model. This observation is highly surprising since deltas of GC model are obtained within the idiosyncratic framework, and one would naturally expect that those deltas will be closer to the corresponding dynamic idiosyncratic values. We have checked that, in the regime of flat correlations where the GC model represents a consistent static model, the results of GC model are, indeed, closer to the idiosyncratic deltas of the dynamic model. This can be seen from Fig. 2 showing idiosyncratic tranche deltas as functions of maturity obtained within the two models in the case of an artificial setup where tranche quotes are generated via GC copula model with flat correlations. However, when significant base correlation skew is present, the situation changes to the opposite and results of GC model get closer to the dynamic values obtained within the contagious approach. Thus, the inconsistency of the base correlation framework causes deltas of GC model to deviate significantly from those obtained within the consistent idiosyncratic dynamic framework and, coincidentally, makes them closer to the values obtained with inclusion of default contagion effect.

Deltas obtained in the heterogeneous case (see Table 4) turn out to be relatively close to the corresponding homogeneous values only for assets with moderate spreads, close to the spread of the index. While all three methods predict deltas to be rather sensitive to the values of the underlying spreads, results obtained within the dynamic contagious method are significantly more uniform over portfolio members. Such smoothing can be intuitively expected to be caused by the default contagion effect. It is important to note that GC model predicts negative deltas for the senior tranche with respect to the asset with low default probability maturing in 5 and 7 years. Neither dynamic method shows this problem. In the case of an asset with high default probability, the two idiosyncratic methods (GC and dynamic) predict very low deltas for senior and supersenior tranches. In the context of the GC model, this effect is well known and can be explained as follows: Under the condition that the portfolio loss has reached senior tranches, the probability of an unreliable asset to survive is close to zero and small changes of its spread become irrelevant. This consideration, however, is of an idiosyncratic nature because we assume that changing the spread of an asset will not affect the spreads of other assets. Thus, one expects that contagion effects will tend to smear this effect out, which is supported by our results: Deltas of senior tranches with respect to the high-spread asset are not small and are significantly larger than results obtained within the idiosyncratic framework.

6 Portfolio with heterogeneous recovery coefficients

Throughout this paper, we have considered a simplified case of a portfolio with homogeneous recovery coefficients. Here, we present a generalization of the developed method to fully heterogeneous portfolios. One of the ways to proceed would be to modify the model making the asset default intensities to be deterministic functions of the portfolio loss. Alternatively, one can try to recover the portfolio loss distribution, making no changes to the existing scheme. While the first way is feasible, it has two disadvantages. First, the loss variable requires a finer grid than that of the discrete number of defaults, and this will result in a significant decrease in the numerical efficiency. The second problem is that the tranche pricing formulas also include the amortization due to recovery, which, to find, requires the knowledge of the distributions of the portfolio recovery (for details see, for example, Ref. [4]) that would be difficult to obtain if the state variable is chosen to be the portfolio loss. The contribution to the tranche value from this mechanism is not significant, nevertheless, the inclusion of this mechanism is a requirement for a model to be used in the industry applications. For these reasons, we prefer to choose the second way to proceed.

In the spirit of the developed semi-analytic approach, we will make use of the fact that the number of assets in the portfolio is large. In this limit, the intensity of the transition with a given loss change, l, conditioned on the number of defaults, N, can be written as

$$t(l, N, t) = \frac{1}{P_B(N, t)} \sum_{k=1}^{N_0} \lambda_k(N, t)\, P_k(N, t)\, \delta(l - h_k), \tag{39}$$

where $\delta(x)$ is the Dirac delta-function and $h_k = A_k(1 - R_k)$ is the loss given default of the $k-$th asset. Equation (39) defines the instantaneous probability of the transition between the two portfolio states characterized by different losses and default numbers. Namely, given that, at time t, the portfolio loss is L, and the number of defaults is N, the generator $t(l, N, t)$ represents the transition $(L, N) \to (L + l, N + 1)$. Thereby, one can write the Focker-Planck equation on the joint probability distribution of the loss and number of defaults, $P(L, N, t)$, as

$$\frac{d}{dt} P(L, N, t) = \int dl \Big(t(l, N - 1, t)\, P(L - l, N - 1, t) - t(l, N, t)\, P(L, N, t) \Big). \tag{40}$$

Given the joint distribution, $P(L, N, t)$, the loss distribution function, $P(L, t)$ can be found as

$$P(L, t) = \sum_{N=0}^{N_0} P(L, N, t). \tag{41}$$

Equations (39, 40) allow construction of the joint probability distribution $P(L, N, t)$ out of the probabilities $P_k(N, t)$. The probability distribution of the loss defining the spreads of CDO tranches can then be obtained from Eq. (41). Numerical solution of Eq. (40) requires discretization of the loss variable. Computationally expensive convolution operation can be bypassed by turning to the Fourier representation. Amortization due to recovery can be found in a similar way, changing the loss variable to recovery and setting $1 - R_i \to R_i$.

Finally, we comment on the parametrization of the surface $Y(N, t)$ in the case of heterogeneous recoveries. In Section 4.2, we presented the scheme that uses linear interpolation in the loss. This scheme used the simple relation between the portfolio loss and number of defaults, $L = Nh$, which is not valid in the case of heterogeneous recoveries. To resolve this problem, one can introduce the expected loss conditioned on the number of defaults,

$$\bar{L}(N) = \mathrm{E}\,[\,L(t)\,|\,N(t) = N\,], \tag{42}$$

that can be easily found from the joint default distribution function $P(L, N, t)$. Parametrization of $Y(N, t)$ can then be done exactly as in Section 4.2 with the substitution $hN \to \bar{L}(N)$.

7 Markovian projection onto default contagion model

The model (1), as any model of the default contagion framework, can hardly provide an adequate description of the stochastic market fluctuations of CDS spreads (the market risk) that are very significant even in the absence of defaults. In this respect, the default contagion framework can be contrasted with the double stochastic approach. Consider, for example, the model used by Eckner (2007) which defines single asset intensities in the form

$$\lambda_k(t) = X_k(t) + a_k Y(t), \tag{43}$$

where $X_k(t)$ and $Y(t)$ are, respectively, stochastic idiosyncratic and systemic components, each following Cox-Ingersoll-Ross (CIR) dynamics with jumps. Model (43) captures the market risk, but it does not include explicitly the default contagion effect. The market risk is important for pricing dynamics-sensitive instruments, such as, for example, options on tranches. Sensitivity of the option value to the stochastic dynamics of the default intensity was confirmed within the models of the top-down framework in works by Arnsdorf & Halperin (2007) and Lopatin & Misirpashaev (2008).

In this section, we show that double stochastic and contagion models are not fundamentally different as long as one deals with instruments whose values can be expressed completely through the full probability density of defaults $p(\mathbf{n}, t)$. In this case, the division of stochastic dynamics of default intensities into the "market" and "contagious" is to some extent illusive in the sense that only additional, more specific, market information would allow distinguishing between the two. Below, we show that the default contagion model can be viewed as a reduced form of a model with default intensities given by adaptive stochastic processes. This procedure can be viewed as a multidimensional generalization of the Markovian projection technique used by Lopatin & Misirpashaev (2008). The same multi-dimensional extension was recently used by Giesecke *at al* (2009) in a context of developing of an efficient simulation method. The most famous example of using the Markovian projection in finance is the projection of the stochastic volatility model onto the Dupire (1994) local volatility model. This projection was first suggested and proved by Gyöngy (1986) in no relation to finance. Detailed discussion of applications of Gyöngy's lemma is given by Piterbarg (2007) [see also Antonov *at el* (2007) for an example of a multidimensional Markovian projection].

In the most general form, the default contagion model defines the default intensities of portfolio members as deterministic functions of the current portfolio state,

$$\lambda_k(t) = \lambda_k(\mathbf{n}, t). \tag{44}$$

Because the model is Markovian, the full basket probability density $p(\mathbf{n}, \mathbf{t})$ obeys the Focker-Planck equation

$$\frac{d}{dt}\, p(\mathbf{n}, t) = \sum_k \hat{\mathcal{L}}_k \lambda_k(\mathbf{n}, t)\, p(\mathbf{n}, t) - \lambda_k(\mathbf{n}, t)\, p(\mathbf{n}, t), \tag{45}$$

where the action of the operator $\hat{\mathcal{L}}_k$ on an arbitrary function $f(\mathbf{n})$ is defined as

$$\begin{aligned} &\hat{\mathcal{L}}_k\ f(n_1,\, \dots, n_k = 1,\, \dots, n_{N_0}) = f(n_1,\, \dots, n_k = 0,\, \dots, n_{N_0}), \\ &\hat{\mathcal{L}}_k\ f(n_1,\, \dots, n_k = 0,\, \dots, n_{N_0}) = 0. \end{aligned} \tag{46}$$

The operator $\hat{\mathcal{L}}_k$ "lowers" the default indicator of the $k-$th asset. It is also assumed in Equation (45) that the default intensity of a defaulted asset is zero:

$$\lambda_k(n_1,\, \dots, n_k = 1,\, \dots, n_{N_0}, t) = 0. \tag{47}$$

Now, let us consider the double stochastic model of a general kind with the default intensity of the $k-$th asset being given by an arbitrary adaptive stochastic process $\lambda_k^a(t)$. This model will reproduce the same default probability density $p(\mathbf{n}, t)$ as model (44) as long as

$$\lambda_k(\mathbf{n}, t) = \mathrm{E}\,[\, \lambda_k^a(t) \,|\, \mathbf{n}(t) = \mathbf{n}\,]. \tag{48}$$

To prove this statement, let us write the probability density $p(\mathbf{n}, t)$ at time $t + \delta t$,

$$p(\mathbf{m}, t + \delta t) = \mathrm{P}[\mathbf{n}(t + \delta t) = \mathbf{m}] = \sum_{\mathbf{n}} \mathrm{P}[\, \mathbf{n}(t + \delta t) = \mathbf{m} \,|\, \mathbf{n}(t) = \mathbf{n}\,]\, \mathrm{P}[\mathbf{n}(t) = \mathbf{n}], \tag{49}$$

and consider transition elements

$$A(\mathbf{m}, \mathbf{n}, t, \delta t) = \mathrm{P}[\, \mathbf{n}(t + \delta t) = \mathbf{m} \,|\, \mathbf{n}(t) = \mathbf{n}\,], \tag{50}$$

in the limit $\delta t \to 0$, keeping only terms of zero and linear order in δt. It is clear that, in this limit, one can leave only elements with $\mathbf{m} = \mathbf{n}$ and $\mathbf{m} = \mathbf{n} + \mathbf{e}_k,\ n_k = 0$, where the vector $\mathbf{e}_k$ has k-th element set to unity, all others being zeros. In the latter case,

$$\lim_{\delta t \to +0} \frac{1}{\delta t} A(\mathbf{n} + \mathbf{e}_k, \mathbf{n}, t, \delta t) = \lim_{\delta t \to +0} \frac{1}{\delta t} \mathrm{P}[\, \mathbf{n}(t + \delta t) = \mathbf{n} + \mathbf{e}_k \,|\, \mathbf{n}(t) = \mathbf{n}\,] = \mathrm{E}\,[\, \lambda_k^a(t) \,|\, \mathbf{n}(t) = \mathbf{n}\,]. \tag{51}$$

From condition $\sum_{\mathbf{m}} A(\mathbf{m}, \mathbf{n}, t, \delta t) = 1$, we find the diagonal elements

$$A(\mathbf{n}, \mathbf{n}, t, \delta t) = 1 - \sum_{\mathbf{m} \neq \mathbf{n}} A(\mathbf{m}, \mathbf{n}, t, \delta t) = 1 - \delta t \sum_k 1_{n_k = 0}\, \mathrm{E}\,[\, \lambda_k^a(t) \,|\, \mathbf{n}(t) = \mathbf{n}\,] + O(\delta t^2). \tag{52}$$

Inserting expressions for matrix elements (51, 52) into Eq. (49) and taking the limit $\delta t \to 0$, we reproduce Eq. (45) with $\lambda_k(\mathbf{n}, t)$ given by conditional expectation in accordance with Eq. (48).

The presented Markovian projection technique provides a link between the double stochastic and contagion models by showing that contagion effect can be "generated" via projecting the stochastic intensity on the portfolio state. This, of course, does not mean that any two models of the two frameworks are guaranteed to give the same or close results for the tranche hedge ratios. Indeed, starting, for example, from the model (43) and constructing the corresponding Markovian contagion model via Eq. (48), one would hardly obtain the deterministic intensities fitting the specific form (1). Also, as we discussed in Section 5.2, the derivation of hedge ratios relies on the fixing of certain model parameters at the stage of recalibrating the model to the perturbed survival probabilities. This may also lead to discrepancies in the results obtained within the two models. Indeed, keeping certain parameters in model (43) constant, in general, would impose a rather complicated constraint on the effective intensities of the default contagion model.

8 Stochastic recovery coefficients

It is established empirically that asset recoveries are negatively correlated with the rate of defaults (see, for example, Hamilton *et al*, 2005). Recent credit crisis has fully revealed the importance of this effect. During this turbulent time period spreads of senior tranches have widened so much that often they could not be fitted via the industry standard, Gaussian Copula model under assumption of constant recovery coefficients (base correlations were hitting the unity bound). An extension of the Gaussian Copula model allowing for inclusion of stochastic, factor correlated, recoveries was developed well before the credit crisis by Andersen & Sidenius (2004). Amraoui and Hitier (2008) have recently suggested a particular form of the dependence of recovery coefficients on factor that automatically respects the consistency with static values of CDS recoveries. Inglis *et al* (2009) use bivariate recovery distribution extending the static version of Inglis & Lipton model. Hull & White (2006) include correlations between the recoveries and default intensity in order to improve the quality of calibration of their implied copula model.

The simplest way to include the effect of negative correlations between recoveries and intensity of defaults in our model is too assume that an asset recovery is a deterministic function of its default intensity. So, the loss given default (LGD) of the $k-$th asset, $L_k = 1 - R_k$, (L_k is in units of asset notional) has a form

$$L_k(t) = f_k(\lambda_k(t)). \tag{53}$$

Prior to choosing a particular form for the dependence $f_k(\lambda)$, we note that a special care should be taken to ensure consistency between Eq. (53) and static recovery coefficients, R_k, used on the stage of obtaining the survival probabilities, $p_k(t)$, from the market CDS spreads. To remind, CDS value is fully determined by the asset survival probability, $p_k(t)$, and expected loss $l_k(t)$. Stripping the survival curve one usually makes an assumption of constant recovery coefficients, or, equivalently, that an asset LGD defined as

$$L_k = \frac{l_k(t)}{p_k(t)} \tag{54}$$

is constant. We note that the realized values of asset losses may be (and in realty certainly are) stochastic, yet, in order to strip a CDS, all one needs is the effective static LGD defined according to Eq. (54). Thus, one has to ensure that stochastic LGD given by Eq. (53) is consistent with the static LGD (54). This can be guaranteed via imposing requirement

$$L_k = \frac{\mathrm{E}[\,L_k(t)\lambda_k(t)\,]}{\mathrm{E}[\,\lambda_k(t)\,]}. \tag{55}$$

In the case of the present model this general consistency requirement becomes

$$L_k = \frac{1}{p_k(t)} \sum_{N=0}^{N_0-1} f_k(\lambda_k(N,t))\,\lambda_k(N,t)\,P_k(N,t). \tag{56}$$

Now we turn to the specification of the function $f_k(\lambda)$. Following Hull & White (2006) we choose the linear dependence of the loss given default on intensity restricted by upper and lower bounds, $L_k^{\min}$ and $L_k^{\max}$, respectively. While, the natural values for these bounds are $L_k^{\min} = 0$ and $L_k^{\max} = 1$, we prefer to keep them arbitrary satisfying $0 \leq L_k^{\min} \leq L_k^{\max} \leq 1$. Thus, we assume

$$f_k(\lambda) = B(a_k + \theta_k\lambda, L_k^{\min}, L_k^{\max}) \tag{57}$$

where $B(x,a,b)$ is

$$B(x,a,b) = \begin{cases} b, & x > b \\ x, & a \leq x \leq b \\ a, & x < a. \end{cases} \tag{58}$$

To make Eq.(57) consistent with constraint (55) we will assume that the slopes θ_k and bound values $L_k^{\min}, L_k^{\max}$ are fixed model parameters, while the values of a_k are chosen such that Eq.(56) is satisfied at all times. We note that, in general, it makes the coefficients a_k to be time dependent even when the input values $(\theta_k, L_k^{\min}, L_k^{\max})$ are constant. One can see that changing a_k one shifts the function $f_k(\lambda)$ along $\lambda-$axis such that $f_k(\lambda)$ takes the values $L_k^{\min}$, $L_k^{\max}$ in the limits $a_k \to -\infty$, $a_k \to \infty$, respectively. This guarantees the existence of the solution of Eq.(56) as long as the loss bounds include the static value of LGD, $L_k^{\min} < L_k < L_k^{\max}$.

Numerical implementation of the above described procedure suites well the forward calibration scheme of Section 4. At each integration time step, t_i, Eq.(56) has to be solved numerically for the values of $a_k(t_i)$. Piece-linear form of the dependence in Eq. (57) allows to find the root within just a few iterations such that numerical overhead due to this procedure turners out to be not significant. In many cases the overall calibration procedure may become even faster because negative correlations between the recoveries and default intensity often result in the reduction of the basket default intensity, which in tern allows to choose a smaller time steps in integration. Table 5 shows the results of model calibration to CDX 12 index tranches quoted on June 19, 2009. Parameters that set correlated recoveries in Eq.(57) were $\theta_k = 2, L_k^{\min} = 0.2, L_k^{\max} = 1$. For this market data the model could not be calibrated without inclusion of correlated recovery with acceptable accuracy.

9 Conclusion

We presented a simple bottom-up credit model that can be calibrated simultaneously to CDO tranches and to the spreads of underlying CDSs. The model is solved within the semi-analytic forward induction method that allows reducing the original multidimensional problem to a system of coupled one-dimensional Focker-Planck equations and is justified when the number of assets in the portfolio is large. While in the present paper we mostly concentrated on the case of homogeneous recovery coefficients, the method can be extended to handle heterogeneous recoveries as well.

We used two different methods for calculating tranche hedge ratios with respect to market changes of spreads of underlying CDSs. The first method takes into account that perturbing the survival probability of an asset affects the dynamics of other portfolio members; the second deliberately disregards this effect. We found that the two methods, in general, produce rather different results, supporting the importance of inclusion of the default contagion effect. Surprisingly, we found that, in the homogeneous portfolio case, deltas calculated via the first method are close to the corresponding values obtained within the GC model. Even in the heterogeneous case, the results are still fairly close for those portfolio members that have spreads close to the spread of the index. In general, the tranche deltas obtained within the contagion method are more homogeneous over portfolio members than those obtained within the idiosyncratic scheme.

10 Acknowledgements

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Appendices

A Derivation of the Focker-Planck equation (3)

We will derive Eq. (3) for a default contagion model of a general kind defined in Eq. (44). In this case, the default intensity $\lambda_k(N,t)$ entering Eq. (3) should be understood as the default intensity of the k-th asset, conditioned on the total number of defaults accumulated in the portfolio,

$$\lambda_k(N,t) = \frac{\sum_{\mathbf{n}} \lambda_k(\mathbf{n},t)\; p(\mathbf{n},t)\, \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+n_2+\ldots+n_{N_0}=N}}{P_k(N,t)}. \tag{59}$$

This equation turns into identity when $\lambda_k(\mathbf{n},t)$ depends on the current portfolio state through the total number of defaults, N, due to the definition of $P_k(N,t)$ given by Eq. (10).

Similarly, based on the definition (5), we can express $\Lambda_k(N,t)$ through the full probability density $p(\mathbf{n},t)$ and intensity of defaults (44) as

$$\Lambda_k(N,t) = \frac{\sum_{\mathbf{n}} \sum_{m;\, m\neq k} \lambda_m(\mathbf{n},t)\; p(\mathbf{n},t)\, \mathbf{1}_{n_m=0}\, \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+n_2+\ldots+n_{N_0}=N}}{P_k(N,t)}. \tag{60}$$

To derive equation (3), we multiply both sides of Eq. (45) by $\mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+n_2+\ldots+n_{N_0}=N}$ and sum over all realization of defaults:

$$\frac{d}{dt}\, P_k(N,t) = \sum_{\mathbf{n}} \sum_{m} \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+\ldots+n_{N_0}=N} \left(\hat{\mathcal{L}}_m \lambda_m(\mathbf{n},t)\, p(\mathbf{n},t) - \lambda_m(\mathbf{n},t)\, p(\mathbf{n},t) \right). \tag{61}$$

The contribution from the first term in the brackets in Eq. (61) can be reduced to first term in the r.h.s. of Eq. (3) as

$$
\begin{aligned}
& \sum_{\mathbf{n}} \sum_{m} \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+\ldots+n_{N_0}=N}\, \hat{\mathcal{L}}_m\, \lambda_m(\mathbf{n},t)\, p(\mathbf{n},t) \\
= & \sum_{\mathbf{n}} \sum_{m;\, m\neq k} \hat{\mathcal{L}}_m\, \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+\ldots+n_{N_0}+1=N}\, \lambda_m(\mathbf{n},t)\, p(\mathbf{n},t) \\
= & \sum_{\mathbf{n}} \sum_{m;\, m\neq k} \mathbf{1}_{n_m=0}\, \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_1+\ldots+n_{N_0}+1=N}\, \lambda_m(\mathbf{n},t)\, p(\mathbf{n},t) \\
= & \Lambda_k(N-1,t) P_k(N-1,t). \qquad (62)
\end{aligned}
$$

The contribution of the second term in the brackets in Eq. (61) brings the term $(\Lambda_k(N,t) + \lambda_k(N,t))P_k(N,t)$ of Eq. (3). This can be observed directly by summing Eqs. (59, 60).

B Self-consistency criterion for the semi-analytic approximation

In this Appendix, we formulate the criterion that guarantees the self-consistency of the semi-analytic approximation. Consider a general approximation for $\Lambda_k(N,t)$,

$$\Lambda_k(N,t) = \Lambda_B(N,t) + \Lambda'_k(N,t), \qquad (63)$$

not specifying the form of the correction $\Lambda'_k(N,t)$. Functions $P_B(N,t)$ and $\Lambda_B(N,t)$ obtained via solution of Eqs. (3, 4, 9, 12) will satisfy Eq. (13) for any $\Lambda'_k(N,t)$ that obeys the following condition,

$$\sum_{k=1}^{N_0} \Lambda'_k\, P_k(N,0) = -P_B(N,t)\Lambda_B(N,t). \qquad (64)$$

First, we will show that criterion (64) is the valid identity. Inserting $\Lambda'_k(N,t) = \Lambda_k(N,t) - \Lambda_B(N,t)$ into the l.h.s. of Eq. (64) and then using Eqs. (9, 12), we write condition (64) in the equivalent form,

$$\sum_{k} \Lambda_k(N,t)P_k(N,t) = (N_0 - N - 1) \sum_{k=1}^{N_0} \lambda_k(N,t)P_k(N,t). \qquad (65)$$

The validity of this equation can be proven analogously to the derivation of Eq. (9). The l.h.s. of Eq. (65) can be written in terms of the full probability density $p(\mathbf{n},t)$ as

$$\sum_{p,\,k,\,k\neq p} \sum_{\mathbf{n}} \lambda_p(N,t)\; p(\mathbf{n},t)\, \mathbf{1}_{n_k=0}\, \mathbf{1}_{n_p=0}\, \mathbf{1}_{n_1+n_2+\ldots+n_{N_0}=N}, \qquad (66)$$

where summation over indices p and k goes from 1 to N_0 and the default indicators take values $n_k = 0, 1$. The r.h.s of Eq. (65) can be written as

$$(N_0 - N - 1) \sum_{p=1}^{N_0} \sum_{\mathbf{n}} \lambda_p(N,t)\; p(\mathbf{n},t)\, \mathbf{1}_{n_p=0}\, \mathbf{1}_{n_1+n_2+\ldots+n_{N_0}=N}. \qquad (67)$$

Equality of Eqs. (66, 67) can be seen by observing that, for a fixed state vector $\mathbf{n}$ and index p, the sum over k in Eq. (66) reduces to multiplication by the factor $N_0 - N - 1$.

Now, we will show that Eq. (64) guarantees the self consistency of the semi-analytic approach. We will proceed analogously to the corresponding consideration in Section 3.1. Taking the sum of both sides of Eq. (3) over k and using Eq. (9), we obtain

$$\begin{aligned}(N_0 - N)\frac{d}{dt}P_B(N,t) &= -\Lambda_B(N,t)P_B(N,t) \\ &+ \sum_{k=1}^{N_0}\Lambda_k(N-1,t)P_k(N-1,t) - \Lambda_k(N,t)P_k(N,t). \end{aligned} \tag{68}$$

One can easily show that this equation can be brought to the form of Eq. (13) under the assumption that Eq. (64) is valid.

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	5y		7y		10y	
	Market	Model	Market	Model	Market	Model
0-3%	34.165	34.2	40.04	40.14	45	45.19
3-6%	370.4	370.9	462.9	460.8	591.8	588.2
6-9%	244.4	245.4	276	278.7	335.7	338.2
9-12%	167.5	166.3	189.7	189.4	218.6	217.6
12-22%	82.4	86	93.2	92.9	105.7	106.6
22-100%	32.2	30	36.6	35	38	36.8

Table 1: Model and market CDO spreads in the homogeneous portfolio case.

	5y		7y		10y	
	Market	Model	Market	Model	Market	Model
0-3%	34.165	34.2	40.04	40.09	45	45.19
3-6%	370.4	370.9	462.9	460	591.8	588.3
6-9%	244.4	245.4	276	277.6	335.7	338.4
9-12%	167.5	168.1	189.7	188.6	218.6	217.6
12-22%	82.4	85.4	93.2	94.3	105.7	107.1
22-100%	32.2	29.4	36.6	34.7	38	36.6

Table 2: Model and market CDO spreads in the heterogeneous portfolio case.

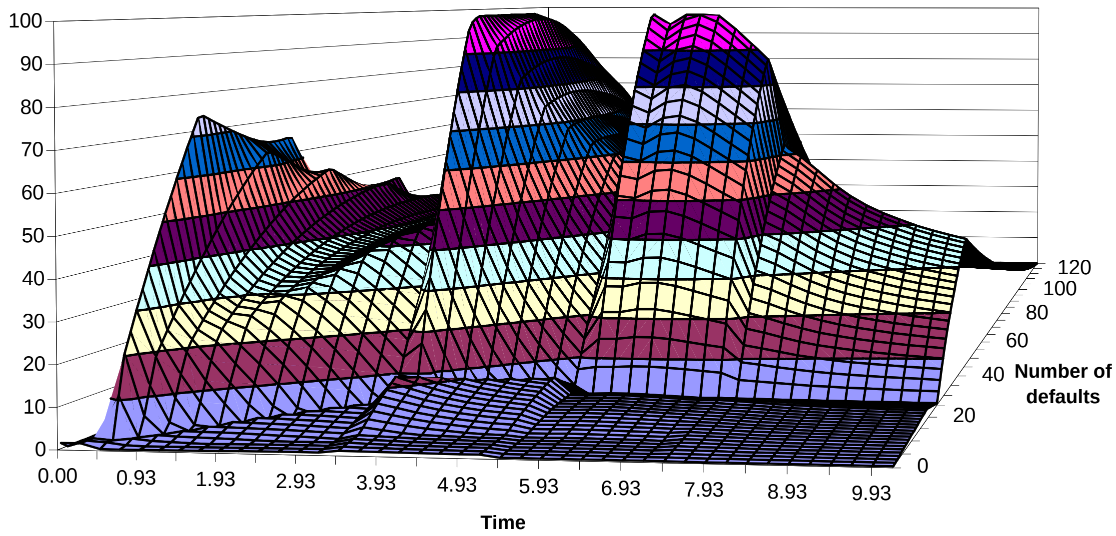

Figure 1: Basket default intensity, $\Lambda_B(N, t)$, in the heterogeneous portfolio case.

Maturity:	5y			7y			10y		
Method:	DC	DI	GC	DC	DI	GC	DC	DI	GC
CDS spreads:	85			91			94.9		
0-3%	0.2441	0.4408	0.2327	0.1859	0.3372	0.1800	0.1483	0.2270	0.1356
3-6%	0.1407	0.1420	0.1456	0.1562	0.2108	0.1516	0.1635	0.2445	0.1529
6-9%	0.0931	0.0848	0.0989	0.1024	0.1033	0.1035	0.1163	0.1528	0.1141
9-12%	0.0678	0.1310	0.0720	0.0703	0.1565	0.0768	0.0804	0.1689	0.0851
12-22%	0.1287	0.0854	0.1346	0.1298	0.0828	0.1465	0.1450	0.1117	0.1650
22-100%	0.3486	0.1516	0.3427	0.3864	0.1639	0.3749	0.3930	0.1798	0.3937

Table 3: Deltas of tranches obtained in dynamic model via contagious (DC) and idiosyncratic (DI) methods are presented, along with corresponding results calculated via the static Gaussian Copula (GC) model in the homogeneous portfolio case.

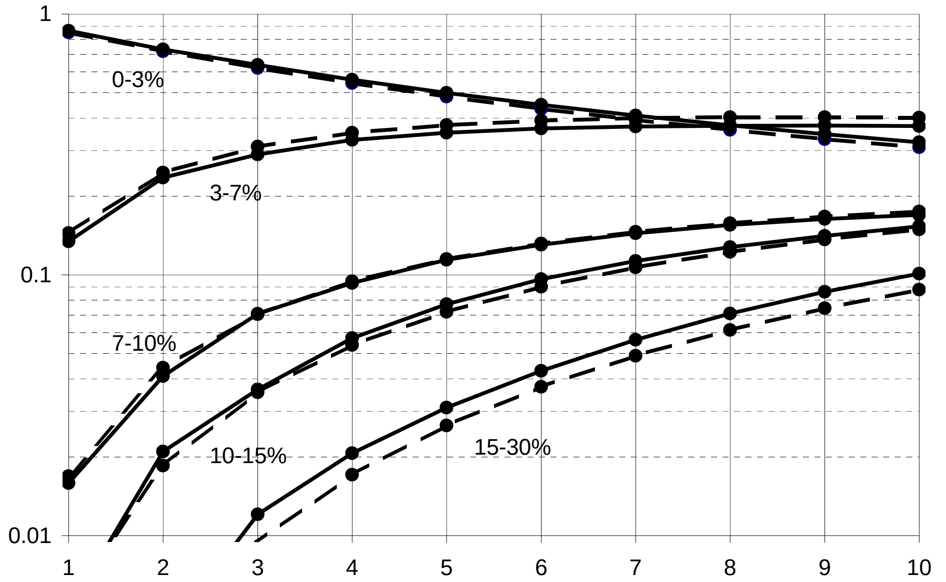

Figure 2: Tranche deltas as functions of maturity in the case of a simple artificial setup where tranche quotes are generated by the Gaussian Copula model with constant correlations set to 0.2. Tranche attachment points were chosen to be $0, 3, 7, 10, 15, 30\%$, and a fully homogeneous portfolio was considered with CDS spreads set to $50\,bp$ and recovery coefficients 40%. Solid lines represent the values obtained via the dynamic idiosyncratic method (the dynamic model was calibrated to the tranche quotes generated by the GC model) while the dashed lines show results of the CG model.

Maturity:	5y			7y			10y		
Method:	DC	DI	GC	DC	DI	GC	DC	DI	GC
CDS spreads:	29.3			33.4			32.4		
0-3%	0.2404	0.1740	0.1064	0.1613	0.1201	0.0753	0.1010	0.0835	0.0512
3-6%	0.1530	0.0570	0.0540	0.1548	0.0722	0.0595	0.1429	0.0846	0.0617
6-9%	0.1006	0.0292	0.0420	0.1072	0.0366	0.0419	0.1180	0.0529	0.0467
9-12%	0.0712	0.0653	0.0290	0.0749	0.0569	0.0296	0.0886	0.0657	0.0321
12-22%	0.1303	0.0275	-0.0035	0.1449	0.0345	-0.0031	0.1650	0.0425	0.0061
22-100%	0.3468	0.6740	0.7961	0.4144	0.7141	0.8248	0.4729	0.7240	0.8440
CDS spreads:	90.7			97			101.9		
0-3%	0.2571	0.4583	0.2064	0.1823	0.3449	0.1572	0.1347	0.2282	0.1165
3-6%	0.1553	0.1550	0.1333	0.1650	0.2199	0.1387	0.1626	0.2518	0.1417
6-9%	0.0994	0.0771	0.0968	0.1119	0.1094	0.0998	0.1217	0.1564	0.1104
9-12%	0.0679	0.1766	0.0734	0.0757	0.1594	0.0771	0.0864	0.1830	0.0853
12-22%	0.1224	0.0691	0.1483	0.1303	0.0944	0.1595	0.1524	0.1092	0.1786
22-100%	0.3211	0.1031	0.3657	0.3677	0.1306	0.3972	0.3893	0.1601	0.4075
CDS spreads:	200			207.5			213		
0-3%	0.2931	0.5560	0.3200	0.2392	0.4335	0.2577	0.1940	0.2975	0.2033
3-6%	0.1560	0.1767	0.2038	0.1828	0.2542	0.2131	0.1933	0.2947	0.2125
6-9%	0.0929	0.0759	0.1264	0.1117	0.1120	0.1372	0.1278	0.1660	0.1504
9-12%	0.0584	0.1440	0.0878	0.0693	0.1402	0.0973	0.0824	0.1695	0.1083
12-22%	0.1011	0.0520	0.1720	0.0956	0.0732	0.1960	0.1073	0.0889	0.2158
22-100%	0.2878	0.0185	0.0956	0.2893	0.0176	0.1046	0.2852	0.0294	0.1199

Table 4: Deltas of tranches obtained in the dynamic model via contagious (DC) and idiosyncratic (DI) methods are presented, along with corresponding results obtained via the static Gaussian Copula (GC) model in the heterogenous portfolio case. Three representative members with low, moderate, and high spreads were chosen from a portfolio of 125 names.

	5y		7y		10y	
	Market	Model	Market	Model	Market	Model
0-3%	63.82	63.82	68.90	68.91	73.30	73.30
3-7%	34.18	34.18	40.98	40.97	46.80	46.80
7-10%	14.74	14.74	19.59	19.61	25.34	25.35
10-15%	4.44	4.43	6.23	6.21	8.64	8.64
15-30%	-0.69	-0.71	-0.83	-0.79	-0.84	-0.81
30-100%	-2.79	-2.86	-3.50	-3.51	-4.01	-4.10

Table 5: Model fit for CDO tranches in the case of CDX 12 quoted on June 19, 2009. Tranches values are given in percents of notionals; spreads of all tranches are 100 bp. Parameters of stochastic recovery are $\theta = 2, L^{\min} = 0.2, L^{\max} = 1$.